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Quantum Mechanics in Finite Dimensions

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Abstract

This paper contributes to a recent series discussing quantum mechanics defined on a finite-dimensional Hilbert space in which Weyl's commutation relation for unitary operators holds. In an earlier paper, Santhanam and Tekumalla (1976) showed that the commutation relation for hermitian operators with a bounded spectrum tends to Heisenberg's standard canonical commutation relation as the spectrum becomes continuous and the dimension $n \to \infty$. The present paper offers a formulation which is coordinate free in the limit $n \to \infty$ and makes the limiting procedure especially transparent.

Introduction

It is well known that, of two or more hermitian operators which satisfy Heisenberg's (1925) canonical commutation relation (hereafter referred to as CCR), at least one of them necessarily has an unbounded spectrum. The corresponding Hilbert space is therefore infinite dimensional. On the other hand, two hermitian operators defined on a finite-dimensional space do not satisfy the standard CCR. It is thus natural to inquire whether an analogue of the CCR can be found for operators with a discrete bounded spectrum acting on a finite-dimensional vector space. The answer is yes. Weyl (1931) rewrote the CCR in an exponential form using unitary operators, and his formulation permits bounded operators.

Weyl's formulation has provided the basis for a recent series of papers (Santhanam and Tekumalla 1976; Santhanam 1976, 1977) which have discussed quantum mechanics defined on a finite-dimensional Hilbert space. The principal idea was to start with Weyl's (1931) commutation relation for two operators of the Abelian group of unitary rotations in ray space. When the group is continuous, the resulting formulation is identical with that of Heisenberg (1925) but, when the group is discrete or discrete–continuous, further possibilities open up.

Weyl's formulation makes use of unitary operators, and many difficulties which arise from restrictions on the domain do not appear in it. In particular, Santhanam (1977) showed that operators with a discrete bounded spectrum (like the angular momentum operators) satisfy Weyl's commutation relation. An interesting consequence also revealed by Santhanam is that, when properly interpreted, Weyl's commutation relation implies a generalized statistics.

Starting with Weyl's formulation, Santhanam and Tekumalla (1976) calculated the commutation relation between two hermitian operators (defined as the generators of the unitary transformation). They demonstrated that this relation reduces to the standard CCR as the spectrum becomes continuous and the dimension $n \to \infty$.

However, in their derivation, they used an explicit matrix representation for operators satisfying Weyl's commutation relation. In the present paper, we develop a formulation which is coordinate free in the limit and which makes the limiting procedure especially clear.

Quantum Mechanics in Finite Dimensions

In the n-dimensional Euclidean space $\kappa^{(n)}$, we choose the orthonormal basis

$$\psi_r = (0,0,...,1,...,0) \qquad \text{for} \qquad r = 1,2,...,n, \tag{1}$$

where the 1 resides in the rth column of the rth unit vector. Let us now consider the two operators N and U, defined such that

$$N\psi_r = (r-1)\psi_r, \tag{2}$$

$$U\psi_r = \psi_{r+1} \qquad \text{for} \qquad r < n, \tag{3a}$$

$$U\psi_n = \psi_1. \tag{3b}$$

It is trivial to see that in the basis (1) the following matrix representations hold for N and U:

$$N = \operatorname{diag}[0,1,2,...,n-1], \tag{4}$$

$$U_{rs} = \delta_{r,s-1} \qquad \text{for} \qquad r < n, \quad s \leqslant n, \tag{5a}$$

$$U_{ns} = \delta_{1,s} \qquad\qquad s \leqslant n, \tag{5b}$$

where δ_{ij} is the Kronecker delta function. It is also trivial to see that the two selected operators U and N have the following properties:

$$U^n = I, \tag{6a}$$

$$V^n = I, \qquad \text{where} \qquad V = \exp(N\ln\varepsilon), \tag{6b}$$

with $\varepsilon = \exp(2\pi i/n)$ being the nth primitive root of unity.

Furthermore, it is easily seen that N and U jointly satisfy

$$(NU-UN)\psi_r = U\psi_r = (U-n\,UP_n)\psi_r \qquad \text{for} \qquad r < n, \tag{7a}$$

where P_r is the projection into the one-dimensional subspace spanned by ψ_r (for $r = 1,2,...,n$). On the other hand, for $r = n$ we have

$$\begin{aligned}(NU-UN)\psi_n &= N\psi_1 - (n-1)U\psi_n = U\psi_n - nU\psi_n \\ &= (U-n\,UP_n)\psi_n.\end{aligned} \tag{7b}$$

Thus, we conclude that the following relation holds:

$$[N,U] = U - n\,UP_n. \tag{8}$$

Now, it is clear from the equations (5) that U is unitary

$$U^\dagger U = UU^\dagger = 1, \tag{9}$$

and so equation (8) can be rewritten in the form

$$U^{-1}NU = N+1-nP_n. \tag{10}$$

The last term on the right-hand side of equation (10) takes care of both the finite nature of the spectrum of N and the cyclic nature of U as defined in (5). Since both sides of equation (10) are self adjoint and P_n commutes with N, we can exponentiate to get

$$\exp(-\mathrm{i}t(U^{-1}NU)) = \exp(-\mathrm{i}tN)\exp(-\mathrm{i}t)\exp(\mathrm{i}tnP_n) \quad \text{for} \quad -\infty < t < \infty\,, \tag{11}$$

or

$$\exp(-\mathrm{i}tN)\,U\exp(\mathrm{i}tN) = U\exp(-\mathrm{i}t)\exp(\mathrm{i}tnP_n)\,. \tag{12}$$

Writing

$$t = 2\pi/n \tag{13}$$

and recognizing that

$$\exp(\mathrm{i}2\pi P_n) = 1\,, \tag{14}$$

allows us to reduce equation (12) to

$$\varepsilon\,U\exp(2\pi\mathrm{i}N/n) = \exp(2\pi\mathrm{i}N/n)\,U\,, \tag{15}$$

which is Weyl's commutation relation.

Writing $f(N)$ in terms of the projection operators as

$$f(N) = \sum_{r=1}^{n} f(r-1)\,P_r \tag{16}$$

gives

$$\begin{aligned} f(N)\,UP_n &= \Big(\sum_{r=1}^{n} f(r-1)\,P_r\Big)\,UP_n = \Big(\sum_{r=1}^{n} f(r-1)\,P_r\Big)P_1\,UP_n \\ &= f(0)\,UP_n\,, \end{aligned} \tag{17}$$

where we have used

$$UP_n = P_1\,UP_n\,. \tag{18}$$

Multiplying equation (8) to the left by $f(N)$ and setting

$$A = f(N)\,U \tag{19}$$

we get (using equation 17)

$$[N, A] = A - nf(0)\,UP_n \tag{20}$$

and, for functions for which $f(0) = 0$, we have

$$[N, A] = A\,. \tag{21}$$

Thus we see that when $f(N) = \sqrt{N}$, for instance, the operator A can be interpreted as the creation operator $a^\dagger$, and in this case we have

$$[N, a^\dagger] = a^\dagger\,. \tag{22}$$

When interpreted properly, this relation has been shown to imply a generalized statistics (Santhanam 1977). Finally we point out that another function for which $f(0) = 0$ is

$$f(N) = \sin(2\pi N/n). \tag{23}$$

Limiting Procedure

So far we have looked at $\kappa^{(n)}$ for finite n. To see what happens in the limit as $n \to \infty$, we imbed $\kappa^{(n)}$ in an infinite-dimensional κ with basis ψ_r $(r = 1, 2, ..., \infty)$. Writing $N^{(n)}$ and $U^{(n)}$ as the imbedded rank n operators given by the equations (4) and (5), we want to show that $N^{(n)}$ and $U^{(n)}$ converge strongly.

Let U be an isometric (one-side unitary) operator on κ defined by

$$U\psi_r = \psi_{r+1}. \tag{24}$$

Then clearly we have

$$(U-U^{(n)})\psi_r = 0 \qquad \text{for} \qquad r < n, \tag{25}$$

and hence

$$(U-U^{(n)}) \sum_{r=1}^{l} \alpha_r \psi_r = 0 \qquad \text{for} \qquad l < n. \tag{26}$$

Therefore

$$(U-U^{(n)})g \to 0 \qquad \text{as} \qquad n \to \infty \tag{27}$$

for all g of the form

$$g = \sum_{r=1}^{l} \alpha_r \psi_r \qquad l \text{ arbitrary}. \tag{28}$$

Such g form a dense set in κ. Since $||U^{(n)}|| = 1 = ||U||$ it follows that $U^{(n)}$ converges strongly (i.e. on all vectors in κ) to U.

Let N be defined by

$$N\psi_r = (r-1)\psi_r \qquad \text{for} \qquad r = 1, 2, ..., \infty, \tag{29}$$

then N is an unbounded operator with the domain

$$\mathscr{D}(N) = \left\{ g \in \kappa \,\middle|\, g = \sum_{k=1}^{\infty} \alpha_k \psi_k; \right.$$
$$\left. \sum_{1}^{\infty} |\alpha_k|^2 < \infty, \qquad \sum_{1}^{\infty} (k-1)^2 |\alpha_k|^2 < \infty \right\}, \tag{30}$$

which is dense. It is trivial to see that $N^{(n)}$ converges strongly to N on $\mathscr{D}(N)$.

For $g \in \mathscr{D}(N)$ we have

$$U^{(n)} g = \sum_{k=1}^{\infty} \alpha_k U^{(n)} \psi_k$$
$$= \sum_{k=1}^{n-1} \alpha_k \psi_{k+1} + \alpha_n \psi_1.$$

The operation of $U^{(n)}$ leaves $\mathscr{D}(N)$ invariant and so too does that of U.

In order to see that

$$n\, U^{(n)} P_n g \tag{31}$$

converges strongly to zero as $n \to \infty$ we restate this expression as follows

$$n\, U^{(n)} P_n g = n\, U^{(n)} P_n \sum_{k=1}^{\infty} \alpha_k \psi_k = n\, \alpha_n \psi_1. \tag{32}$$

As a consequence of equation (32) we obtain

$$||n\,U^{(n)}P_n g|| = |n\alpha_n| \to 0, \tag{33}$$

since we have

$$\sum_{n=0}^{\infty} |n\alpha_n|^2 < \infty$$

from the definition (30) of $\mathscr{D}(N)$, given that $g \in \mathscr{D}(N)$ holds. Combining the results (32) and (33) with equation (8) we see that the commutation relation

$$[N^{(n)}, U^{(n)}]g = (U^{(n)} - n\,U^{(n)}P_n)g \tag{34}$$

converges to

$$[N, U]g = Ug \tag{35}$$

for all $g \in \mathscr{D}(N)$. We note that equation (35) formed the basis for the result obtained by Santhanam (1977) that a generalized statistics implied in finite dimensions goes over to Bose statistics as $n \to \infty$.

Since U is only isometric and not unitary, it is *not* in general possible to write $U = \exp(\mathrm{i}\phi)$. But, assuming that it is possible to do so, we may write

$$U(t) = \exp(-\mathrm{i}tN)\,U\exp(\mathrm{i}tN) \tag{36}$$

to obtain the differential equation

$$\mathrm{d}U(t)/\mathrm{d}t = -\mathrm{i}\,U(t). \tag{37}$$

A solution of this equation is

$$U(t) = U\exp(-\mathrm{i}t) \tag{38}$$

or

$$\exp(-\mathrm{i}tN)\exp(\mathrm{i}\phi)\exp(\mathrm{i}tN) = \exp(\mathrm{i}\phi)\exp(-\mathrm{i}t), \tag{39}$$

which could have been got by taking the limit in equation (12). Now equation (39) implies that

$$\exp(-\mathrm{i}tN)\,\phi\exp(\mathrm{i}tN) = \phi - t, \tag{40}$$

which shows that

$$[N, \phi] = -\mathrm{i}. \tag{41}$$

Because of its lack of generality equation (41) is merely a formal result and is not as important as the general result (35).

The following remarks concerning the approach adopted in this paper are in order. In deriving equation (8) we have made use of only (1), (2) and (3), which are definitely coordinate free. However, in finite space, these equations do imply the matrix representations (4) and (5) for the operators N and U. Therefore, the usefulness of the present discussion lies essentially in considering the limit as $n \to \infty$, when the second term on the right-hand side of equation (8) vanishes and we obtain the standard CCR.

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